

ON THE MOD p^2 DETERMINATION OF $\sum_{k=1}^{p-1} H_k/(k \cdot 2^k)$: ANOTHER PROOF OF A CONJECTURE BY SUN

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ABSTRACT. For a positive integer n let $H_n = \sum_{k=1}^n 1/k$ be the n th harmonic number. Z. W. Sun conjectured that for any prime $p \geq 5$,

$$\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2}.$$

This conjecture is recently confirmed by Z. W. Sun and L. L. Zhao. In this note we give another proof of the above congruence by establishing congruences for all the sums of the form $\sum_{k=1}^{p-1} 2^{\pm k} H_k^r / k^s \pmod{p^{4-r-s}}$ with $(r, s) \in \{(1, 1), (1, 2), (2, 1)\}$.

1. THE MAIN RESULTS

Given positive integers n and m , the *harmonic numbers of order m* are those rational numbers $H_{n,m}$ defined as

$$H_{n,m} = \sum_{k=1}^n \frac{1}{k^m}.$$

For simplicity, we will denote by

$$H_n := H_{n,1} = \sum_{k=1}^n \frac{1}{k}$$

the n th *harmonic number* (in addition, we define $H_0 = 0$).

Recently, Z. W. Sun [9] obtained basic congruences modulo a prime $p \geq 5$ for several sums of terms involving harmonic numbers. In particular, Sun established $\sum_{k=1}^{p-1} H_k^r \pmod{p^{4-r}}$ for $r = 1, 2, 3$. Further generalizations of these congruences have been recently obtained by Tauraso in [12].

Recall that Bernoulli numbers B_0, B_1, B_2, \dots are given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0 \quad (n = 1, 2, 3, \dots).$$

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In this note we establish six congruences involving harmonic numbers contained in the following result.

Theorem 1.1. *Let $p > 5$ be a prime. Then*

$$(1) \quad \sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p(2)^2 + \frac{2}{3} p q_p(2)^3 + \frac{p}{12} B_{p-3} \pmod{p^2},$$

$$(2) \quad \sum_{k=1}^{p-1} \frac{2^k H_k}{k^2} \equiv -\frac{1}{3} q_p(2)^3 + \frac{23}{24} B_{p-3} \pmod{p},$$

$$(3) \quad \sum_{k=1}^{p-1} \frac{H_k}{k^2 \cdot 2^k} \equiv \frac{5}{8} B_{p-3} \pmod{p},$$

$$(4) \quad \sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} \equiv -\frac{1}{3} q_p(2)^3 + \frac{11}{24} B_{p-3} \pmod{p},$$

$$(5) \quad \sum_{k=1}^{p-1} \frac{H_k^2}{k \cdot 2^k} \equiv \frac{7}{8} B_{p-3} \pmod{p}$$

and

$$(6) \quad \sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} \equiv -\frac{1}{3} q_p(2)^3 - \frac{25}{24} B_{p-3} \pmod{p}.$$

As an application, we obtain a result obtained quite recently by Z. W. Sun and L. L. Zhao in [10].

Corollary 1.2. ([10, Theorem 1.1]) *Let $p > 5$ be a prime. Then*

$$(7) \quad \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2}$$

and

$$(8) \quad \sum_{k=1}^{p-1} \frac{H_{k,2}}{k \cdot 2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p}.$$

Remark 1.3. The congruence (7) is conjectured by Z. W. Sun in [9, Conjecture 1.1] and quite recently proved by Z. W. Sun and L. L. Zhao in [10]. We point out that Lemma 2.3 from [10] presents the main auxiliary result in the proof of (7) and its proof is based on a polynomial congruence recently obtained by L. L. Zhao and Z. W. Sun in [13, Theorem 1.2]. Moreover, in this proof the authors also use the congruence $\sum_{k=1}^{p-1} H_k/k^2 \equiv B_{p-3} \pmod{p}$ obtained by Sun and Tauraso in [11, the congruence (5.4)].

Notice also that the first congruence in [9, Conjecture 1.1] is also proved by the author of this note in [6, Theorem 1.1 (3)].

Reducing the modulus in (1) we have

Corollary 1.4. *Let $p > 5$ be a prime. Then*

$$(9) \quad \sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p(2)^2 \pmod{p}.$$

This paper is organized as follows. In the next section, using numerous classical and recent combinatorial congruences, we prove the congruence (1). Applying (1) and some auxiliary results, in Section 3 we establish the congruences (2) and (3). Section 4 is devoted to the proof of (4), (5) and (6) based on the previous congruences and an identity for harmonic numbers. As an application, in Section 5 we prove Corollary 1.2 which contains two congruences recently obtained by Z. W. Sun and L. L. Zhao in [10].

2. PROOF OF THE CONGRUENCE (1)

Lemma 2.1. *If $p \geq 3$ is a prime, then*

$$(10) \quad \binom{p-1}{k} \equiv (-1)^k - (-1)^k p H_k + (-1)^k \frac{p^2}{2} (H_k^2 - H_{k,2}) \pmod{p^3}$$

for each $k = 1, 2, \dots, p-1$. In particular, we have

$$(11) \quad \binom{p-1}{k} \equiv (-1)^k - (-1)^k p H_k \pmod{p^2}.$$

Proof. For a fixed $1 \leq k \leq p-1$ we have

$$\begin{aligned} (-1)^k \binom{p-1}{k} &= \prod_{i=1}^k \left(1 - \frac{p}{i}\right) \equiv 1 - \sum_{i=1}^k \frac{p}{i} + \sum_{1 \leq i < j \leq k} \frac{p^2}{ij} \pmod{p^3} \\ &= 1 - p H_k + \frac{p^2}{2} \left(\left(\sum_{i=1}^k \frac{1}{i} \right)^2 - \sum_{i=1}^k \frac{1}{i^2} \right) \\ &= 1 - p H_k + \frac{p^2}{2} (H_k^2 - H_{k,2}) \pmod{p^3}, \end{aligned}$$

whence we have (10). Notice that reducing the modulus into (10) yields (11). \square

Lemma 2.2. *If $p > 3$ is a prime, then*

$$(12) \quad H_{p-1} := \sum_{k=1}^{p-1} \frac{1}{k} \equiv -\frac{p^2}{3} B_{p-3} \pmod{p^3},$$

$$(13) \quad H_{p-1,2} := \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2p}{3} B_{p-3} \pmod{p^2},$$

$$(14) \quad H_{p-1,3} := \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv 0 \pmod{p^2},$$

$$(15) \quad H_{(p-1)/2} := \sum_{k=1}^{(p-1)/2} \frac{1}{k} \equiv -2q_2(p) + pq_2(p)^2 - \frac{2p^2}{3}q_2(p)^3 - \frac{7p^2}{12}B_{p-3} \pmod{p^3},$$

$$(16) \quad H_{(p-1)/2,2} := \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \frac{7p}{3}B_{p-3} \pmod{p^2}$$

and

$$(17) \quad H_{(p-1)/2,3} := \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \equiv -2B_{p-3} \pmod{p}.$$

Proof. The congruence (12) is proved in [5]; see also [7, Theorem 5.1(a)], while (13) is a particular case of [7, Corollary 5.1]. The well known congruence (14) is a particular case of [2, Theorem 3 (b)] and (15) is in fact the congruence (c) in [7, Theorem 5.2]. Further, the congruences (16) and (17) are the congruences (a) with $k = 2$ and (b) with $k = 3$ in [7, Corollary 5.2], respectively. \square

We will also need the following six congruences recently established by Z. H. Sun [8] and Dilcher and Skula [3].

Lemma 2.3. *Let $p > 3$ be a prime. Then*

$$(18) \quad \sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2q_p(2) - \frac{7p^2}{12}B_{p-3} \pmod{p^3},$$

$$(19) \quad \sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -q_p(2)^2 + p \left(\frac{2}{3}q_p(2)^3 + \frac{7}{6}B_{p-3} \right) \pmod{p^2},$$

$$(20) \quad \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv q_p(2) - \frac{p}{2}q_p(2)^2 \pmod{p^2},$$

$$(21) \quad \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv -\frac{1}{2}q_p(2)^2 \pmod{p},$$

$$(22) \quad \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3}q_p(2)^3 - \frac{7}{24}B_{p-3} \pmod{p}$$

and

$$(23) \quad \sum_{k=1}^{p-1} \frac{1}{k^3 \cdot 2^k} \equiv \frac{1}{6} q_p(2)^3 + \frac{7}{48} B_{p-3} \pmod{p}.$$

Proof. The congruences (18)–(21) are in fact the congruences (i)–(iv) in [8, Theorem 4.1]. By the congruence (5) in [3, Theorem 1],

$$\sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{48} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \pmod{p},$$

from which since by (17),

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \equiv -2B_{p-3} \pmod{p}$$

we immediately obtain (22). Finally, (23) follows immediately from (22) by applying the substitution trick $k \mapsto p - k$ and the fact that $2^p \equiv 2 \pmod{p}$ by Fermat little theorem. \square

Lemma 2.4. *Let n be an arbitrary positive integer. Then*

$$(24) \quad \sum_{1 \leq k \leq i \leq n} \frac{2^k - 1}{ki} = \sum_{j=1}^n \frac{1}{j^2} \binom{n}{j}.$$

Proof. Using the well known identities $\sum_{k=j}^i \binom{k-1}{i-1} = \binom{i}{j}$ with $j \leq i$, $\frac{1}{k} \binom{k}{j} = \frac{1}{j} \binom{k-1}{j-1}$ with $j \leq k$, and the fact that $\binom{k}{j} = 0$ when $k < j$, we have

$$\begin{aligned} \sum_{1 \leq k \leq i \leq n} \frac{2^k - 1}{ki} &= \sum_{1 \leq k \leq i \leq n} \frac{(1+1)^k - 1}{ki} = \sum_{1 \leq k \leq i \leq n} \frac{1}{i} \sum_{j=1}^k \frac{1}{k} \binom{k}{j} \\ &= \sum_{1 \leq k \leq i \leq n} \frac{1}{i} \sum_{j=1}^n \frac{1}{j} \binom{k-1}{j-1} = \sum_{j=1}^n \frac{1}{j} \sum_{1 \leq k \leq i \leq n} \frac{1}{i} \binom{k-1}{j-1} \\ &= \sum_{j=1}^n \frac{1}{j} \sum_{j \leq k \leq i \leq n} \frac{1}{i} \binom{k-1}{j-1} = \sum_{j=1}^n \frac{1}{j} \sum_{i=j}^n \frac{1}{i} \sum_{k=j}^i \binom{k-1}{j-1} \\ &= \sum_{j=1}^n \frac{1}{j} \sum_{i=j}^n \frac{1}{i} \binom{i}{j} = \sum_{j=1}^n \frac{1}{j} \sum_{i=j}^n \frac{1}{j} \binom{i-1}{j-1} \\ &= \sum_{j=1}^n \frac{1}{j^2} \sum_{i=j}^n \binom{i-1}{j-1} = \sum_{j=1}^n \frac{1}{j^2} \binom{n}{j}, \end{aligned}$$

as desired. \square

Lemma 2.5. *Let $p > 3$ be a prime. Then*

$$(25) \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{p-1}{k} \equiv \frac{3p}{4} B_{p-3} \pmod{p^2}.$$

Proof. By the congruence (11) of Lemma 2.1, we have

$$(26) \quad \begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{p-1}{k} &\equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} (1 - pH_k) \pmod{p^2} \\ &= \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} - p \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k^2} \pmod{p^2}. \end{aligned}$$

Using (13) and (16) of Lemma 2.2 we have

$$(27) \quad \begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} &= 2 \sum_{\substack{1 \leq j \leq p-1 \\ 2|j}} \frac{1}{j^2} - \sum_{k=1}^{p-1} \frac{1}{k^2} \\ &= \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} - \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{p}{2} B_{p-3} \pmod{p^2}. \end{aligned}$$

Similarly, by (14) and (17) of Lemma 2.2 we have

$$(28) \quad \begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} &= 2 \sum_{\substack{1 \leq j \leq p-1 \\ 2|j}} \frac{1}{j^3} - \sum_{k=1}^{p-1} \frac{1}{k^3} \\ &= \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} - \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -\frac{1}{2} B_{p-3} \pmod{p}. \end{aligned}$$

Since $p \mid H_{p-1}$, it follows that for each $k = 1, 2, \dots, p-1$,

$$(29) \quad H_k = H_{p-1} - \sum_{i=1}^{p-k-1} \frac{1}{p-i} \equiv \sum_{i=1}^{p-k-1} \frac{1}{i} = H_{p-k-1} \pmod{p}.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k^2} &= \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \left(H_{k-1} + \frac{1}{k} \right) \\ &= \sum_{k=1}^{p-1} \frac{(-1)^k H_{k-1}}{k^2} + \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} = \sum_{k=1}^{p-1} \frac{(-1)^{p-k} H_{p-k-1}}{(p-k)^2} + \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \\ &\equiv - \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k^2} + \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \pmod{p} \end{aligned}$$

from which taking (28) we have

$$(30) \quad \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k^2} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \equiv -\frac{1}{4} B_{p-3} \pmod{p}.$$

Finally, substituting (27) and (30) into (26) we obtain (25). \square

Proof of the congruence (1). Observe that the identity (24) of Lemma 2.4 with $n = p - 1$ may be written as

$$(31) \quad \sum_{1 \leq k < i \leq p-1} \frac{2^k}{ki} + \sum_{k=1}^{p-1} \frac{2^k}{k^2} - \sum_{1 \leq k \leq i \leq p-1} \frac{1}{ki} = \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{p-1}{k}.$$

Further, from (12) of Lemma 2.2 we see that $H_{p-1} \equiv 0 \pmod{p^2}$ (the well known Wolstenholme's theorem [1] or [4]), and thus for each $k = 0, 1, 2, \dots, p-2$,

$$\sum_{i=k+1}^{p-1} \frac{1}{i} \equiv -\sum_{i=1}^k \frac{1}{i} = -H_k \pmod{p^2}.$$

Therefore,

$$(32) \quad \sum_{1 \leq k < i \leq p-1} \frac{2^k}{ik} = \sum_{k=1}^{p-1} \frac{2^k}{k} \sum_{i=k+1}^{p-1} \frac{1}{i} \equiv -\sum_{k=1}^{p-1} \frac{2^k H_k}{k} \pmod{p^2}.$$

Further, from the shuffle relation

$$2 \sum_{1 \leq k \leq i \leq p-1} \frac{1}{ki} = \left(\sum_{k=1}^{p-1} \frac{1}{k} \right)^2 + \sum_{k=1}^{p-1} \frac{1}{k^2} = H_{p-1}^2 + H_{p-1,2}$$

by setting the Wolstenholme's congruence $H_{p-1} \equiv 0 \pmod{p^2}$ and (13) of Lemma 2.2, we obtain

$$(33) \quad \sum_{1 \leq k \leq i \leq p-1} \frac{1}{ki} \equiv \frac{p}{3} B_{p-3} \pmod{p^2}.$$

Finally, substituting (25) of Lemma 2.5, (19) of Lemma 2.3, (32) and (33) into the equality (31), we get

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p(2)^2 + \frac{2}{3} p q_p(2)^3 + \frac{p}{12} B_{p-3} \pmod{p^2}$$

which is the desired congruence (1). \square

3. PROOF OF THE CONGRUENCES (2) AND (3)

Lemma 3.1. *Let n be a positive integer. Then*

$$(34) \quad \sum_{k=1}^{n-1} \frac{(-2)^k}{k} \binom{n}{k} = \begin{cases} -2H_{n-1} + H_{(n-1)/2} + \frac{2^n-2}{n} & \text{if } n \text{ is odd} \\ -2H_n + H_{n/2} - \frac{2^n}{n} & \text{if } n \text{ is even.} \end{cases}$$

Proof. In the proof of Lemma 4.1 in [8] it was proved that of each positive odd integer n holds

$$(35) \quad \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{n}{k} x^k = \sum_{k=1}^{n-1} \frac{(1-x)^k - 1}{k} - \frac{1-x^n + (x-1)^n}{n}, \quad x \in \mathbb{R}.$$

Taking $x = 2$ into (35), we obtain

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{(-2)^k}{k} \binom{n}{k} &= \sum_{k=1}^{n-1} \frac{(-1)^k - 1}{k} - \frac{2 - 2^n}{n} \\ &= -2 \sum_{\substack{1 \leq k \leq n-1 \\ k \text{ odd}}} \frac{1}{k} + \frac{2^n - 2}{n} = -2 \left(\sum_{k=1}^{n-1} \frac{1}{k} - \sum_{\substack{1 \leq k \leq n-1 \\ k \text{ even}}} \frac{1}{k} \right) + \frac{2^n - 2}{n} \\ &= -2 \left(H_{n-1} - \frac{1}{2} H_{(n-1)/2} \right) + \frac{2^n - 2}{n} \\ &= -2H_{n-1} + H_{(n-1)/2} + \frac{2^n - 2}{n}. \end{aligned}$$

This proves the first equality in (34).

Now suppose that n is even. Then by the binomial formula, for each $t > 0$ and $x \in \mathbb{R}$, we have

$$(36) \quad \frac{(1-xt)^n - 1}{t} = \sum_{k=1}^n \frac{\binom{n}{k} (-xt)^k}{t} dt = \sum_{k=1}^n \binom{n}{k} (-x)^k t^{k-1}.$$

Since $\int_0^1 t^{k-1} dt = 1/k$, setting $y = 1-xt$ (cf. proof of Lemma 4.1 in [8]) (36) gives

$$\begin{aligned} \sum_{k=1}^n \frac{(-x)^k}{k} \binom{n}{k} &= \int_0^1 \sum_{k=1}^n \binom{n}{k} (-x)^k t^{k-1} dt = \int_0^1 \frac{(1-xt)^n - 1}{t} dt \\ (37) \quad &= -\frac{1}{x} \int_1^{1-x} \frac{x(y^n - 1)}{1-y} dy = \int_1^{1-x} \sum_{k=1}^n y^{k-1} dy \\ &= \sum_{k=1}^n \frac{(1-x)^k - 1}{k}. \end{aligned}$$

Taking $x = 2$ into (37), we obtain

$$\sum_{k=1}^{n-1} \frac{(-2)^k}{k} \binom{n}{k} + \frac{2^n}{n} = -2 \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \frac{1}{k} = -2H_n + H_{n/2}$$

which yields the second identity of (34). \square

Lemma 3.2. *Let $p > 3$ be a prime. Then*

$$(38) \quad \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p}{k} \equiv pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 + \frac{1}{12}p^2B_{p-3} \pmod{p^3}$$

and

$$(39) \quad \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p-1}{k} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 + \frac{1}{12}p^2B_{p-3} \pmod{p^3}.$$

Proof. Setting $n = p$ in the first equality of (34) of Lemma 3.1 and using the congruences (12) and (15) from Lemma 2.2 reduced modulo p^2 , we obtain

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p}{k} &= H_{p-1} - \frac{1}{2}H_{(p-1)/2} + 2q_p(2) \\ &\equiv pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 + \frac{1}{12}p^2B_{p-3} \pmod{p^3}. \end{aligned}$$

This proves (38). Taking $n = p-1$ into the second equality of (34) from Lemma 3.1 and substituting the congruences (12) and (15) from Lemma 2.2 into this, we obtain

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p-1}{k} &= \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p-2}{k} + \frac{2^{p-1}}{p-1} \\ &= -2H_{p-1} + H_{(p-1)/2} \\ &\equiv -2q_p(2) + pq_p(2)^2 - \frac{2p^2}{3}q_p(2)^3 + \frac{1}{12}p^2B_{p-3} \pmod{p^3}. \end{aligned}$$

This is the congruence (39) and the proof is completed. \square

Proof of the congruences (2) and (3). First notice that

$$(40) \quad \begin{aligned} \sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} &= \sum_{k=1}^{p-1} \frac{2^k \left(H_{k-1} + \frac{1}{k}\right)^2}{k} \\ &= \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} + 2 \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k^2} + \sum_{k=1}^{p-1} \frac{2^k}{k^3}. \end{aligned}$$

Further, using (11) of Lemma 2.1 and the identity $\binom{p-1}{k-1} = \frac{k}{p} \binom{p}{k}$, we find that

$$\begin{aligned}
 \sum_{k=1}^{p-1} \frac{2^k p H_{k-1}}{k^2} &\equiv \sum_{k=1}^{p-1} \frac{2^k}{k^2} \left(1 - (-1)^{k-1} \binom{p-1}{k-1} \right) \pmod{p^2} \\
 (41) \quad &= \sum_{k=1}^{p-1} \frac{2^k}{k^2} + \sum_{k=1}^{p-1} \frac{(-2)^k}{k^2} \binom{p-1}{k-1} \\
 &= \sum_{k=1}^{p-1} \frac{2^k}{k^2} + \frac{1}{p} \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p}{k}.
 \end{aligned}$$

Substituting the congruences (19) from Lemma 2.3 and (38) from Lemma 3.2 into (41), we obtain

$$\sum_{k=1}^{p-1} \frac{2^k p H_{k-1}}{k^2} \equiv \frac{5p}{4} B_{p-3} \pmod{p^2},$$

or equivalently,

$$(42) \quad \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k^2} \equiv \frac{5}{4} B_{p-3} \pmod{p}.$$

Now we have

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k^2} = \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k^2} + \sum_{k=1}^{p-1} \frac{2^k}{k^3},$$

whence inserting (42) and (22) from Lemma 2.3 we immediately obtain (2).

Since by (29) $H_{p-k-1} \equiv H_k \pmod{p}$ for each $k = 1, 2, \dots, p-1$ and $2^p \equiv 2 \pmod{p}$, we have

$$\begin{aligned}
 (43) \quad \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k^2} &= \sum_{k=1}^{p-1} \frac{2^{p-k} H_{p-k-1}}{(p-k)^2} \\
 &\equiv \sum_{k=1}^{p-1} \frac{2^{1-k} H_k}{k^2} = 2 \sum_{k=1}^{p-1} \frac{H_k}{k^2 \cdot 2^k} \pmod{p}.
 \end{aligned}$$

Comparing (42) and (43) yields (3). □

4. PROOF OF THE CONGRUENCES (4), (5) AND (6)

Lemma 4.1. *Let n be a positive integer. Then*

$$(44) \quad \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k-1} = \begin{cases} \frac{2^{n-1}(1-n)}{n+1} & \text{if } n \text{ is odd} \\ \frac{(n-1)2^{n-1}+1}{n+1} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Multiplying by $-1/2$ the identity (34) of Lemma 3.1, it becomes

$$(45) \quad \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k} = \begin{cases} H_{n-1} - \frac{1}{2}H_{(n-1)/2} - \frac{2^{n-1}-1}{n} & \text{if } n \text{ is odd} \\ H_n - \frac{1}{2}H_{n/2} + \frac{2^{n-1}}{n} & \text{if } n \text{ is even.} \end{cases}$$

Now the identities $\binom{n}{k-1} = \binom{n+1}{k} - \binom{n}{k}$ and (45) for any odd positive integer n give

$$(46) \quad \begin{aligned} & \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k-1} = \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n+1}{k} - \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k} \\ &= \sum_{k=1}^n \frac{(-2)^{k-1}}{k} \binom{n+1}{k} - \frac{(-2)^{n-1}(n+1)}{n} - \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k} \\ &= \left(H_{n+1} - \frac{1}{2}H_{(n+1)/2} + \frac{2^n}{n+1} \right) - \frac{2^{n-1}(n+1)}{n} \\ & - \left(H_{n-1} - \frac{1}{2}H_{(n-1)/2} - \frac{2^{n-1}-1}{n} \right) \\ &= (H_{n+1} - H_{n-1}) - \frac{1}{2}(H_{(n+1)/2} - H_{(n-1)/2}) + \frac{2^n}{n+1} - \frac{n2^{n-1}+1}{n} \\ &= \left(\frac{1}{n} + \frac{1}{n+1} \right) - \frac{1}{n+1} + \frac{2^n}{n+1} - \frac{n2^{n-1}+1}{n} \\ &= \frac{2^{n-1}(1-n)}{n+1}. \end{aligned}$$

Similarly, using the identities $\binom{n}{k-1} = \binom{n+1}{k} - \binom{n}{k}$ and (45) for even positive integer n we have

$$(47) \quad \begin{aligned} & \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k-1} = \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n+1}{k} - \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k} \\ &= \sum_{k=1}^n \frac{(-2)^{k-1}}{k} \binom{n+1}{k} - \frac{(-2)^{n-1}(n+1)}{n} - \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k} \\ &= \left(H_n - \frac{1}{2}H_{n/2} - \frac{2^{n-1}-1}{n+1} \right) + \frac{2^{n-1}(n+1)}{n} \\ & - \left(H_n - \frac{1}{2}H_{n/2} + \frac{2^{n-1}}{n} \right) \\ &= \frac{1-2^n}{n+1} + 2^{n-1} = \frac{(n-1)2^{n-1}+1}{n+1}. \end{aligned}$$

The equalities (46) and (47) are in fact (44) and the proof is completed. \square

Lemma 4.2. *Let n be an arbitrary positive integer. Then*

$$(48) \quad (-1)^n \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k} 2^k H_k = (2^n - 2) H_{n-1} + H_{[n/2]} + \frac{2^n - 2}{n}$$

where $[x]$ denotes the integer part of x .

Proof. We proceed by induction on n . An immediate computation shows that (48) is satisfied for $n = 1$ and $n = 2$. For every $n = 1, 2, \dots$ put

$$S_n = \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k} 2^k H_k.$$

Then using the identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ we have

$$\begin{aligned} S_{n+1} &= \sum_{k=1}^n (-1)^{k-1} \binom{n+1}{k} 2^k H_k \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n+1}{k} 2^k H_k + (-1)^{n-1} (n+1) 2^n H_n \\ &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} 2^k H_k + \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k-1} 2^k H_k + (-1)^{n-1} (n+1) 2^n H_n \\ &= S_n + 2 \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k-1} 2^{k-1} \left(H_{k-1} + \frac{1}{k} \right) + (-1)^{n-1} (n+1) 2^n H_n \\ &= S_n + 2 \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k-1} 2^{k-1} H_{k-1} + 2 \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} 2^{k-1} \\ (49) \quad &+ (-1)^{n-1} (n+1) 2^n H_n \\ &= S_n - 2 \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k} 2^k H_k - 2(-1)^{n-1} n 2^{n-1} H_{n-1} \\ &\quad + 2 \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} 2^{k-1} + (-1)^{n-1} (n+1) 2^n H_n \\ &= S_n - 2S_n + 2 \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} 2^{k-1} \\ &\quad - 2(-1)^{n-1} n 2^{n-1} \left(H_n - \frac{1}{n} \right) + (-1)^{n-1} (n+1) 2^n H_n \\ &= -S_n + 2 \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} 2^{k-1} + (-1)^{n-1} 2^n (H_n + 1). \end{aligned}$$

Notice that both equalities (44) from Lemma 4.1 for any positive integer n can be written as

$$(50) \quad \sum_{k=1}^{n-1} \frac{(-2)^{k-1}}{k} \binom{n}{k-1} = (-1)^n \frac{(n-1)2^{n-1}}{n+1} + (1 + (-1)^n) \frac{1}{2(n+1)}.$$

Next, substituting (50) into (49) multiplied by $(-1)^{n+1}$, we find that

$$(51) \quad \begin{aligned} & (-1)^{n+1} S_{n+1} \\ &= (-1)^n S_n + 2(-1)^{n+1} \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} 2^{k-1} + 2^n (H_n + 1) \\ &= (-1)^n S_n - 2 \frac{(n-1)2^{n-1}}{n+1} - (1 + (-1)^n) \frac{1}{(n+1)} + 2^n (H_n + 1) \\ &= (-1)^n S_n + \frac{2^{n+1} - 1 - (-1)^n}{n+1} + 2^n H_n. \end{aligned}$$

□

By the induction hypothesis, we have

$$(52) \quad (-1)^n S_n = (-1)^n \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n}{k} 2^k H_k = (2^n - 2)H_{n-1} + H_{[n/2]} + \frac{2^n - 2}{n}$$

which substituting into (51) gives

$$(53) \quad \begin{aligned} & (-1)^{n+1} S_{n+1} \\ &= (2^n - 2)H_{n-1} + H_{[n/2]} + \frac{2^n - 2}{n} + \frac{2^{n+1} - 1 - (-1)^n}{n+1} + 2^n H_n \\ &= (2^n - 2) \left(H_n - \frac{1}{n} \right) + 2^n H_n + H_{[n/2]} + \frac{2^n - 2}{n} + \frac{2^{n+1} - 1 - (-1)^n}{n+1} \\ &= (2^{n+1} - 2)H_n + H_{[n/2]} + \frac{(2^{n+1} - 2) + (1 + (-1)^n)}{n+1} \\ &= (2^{n+1} - 2)H_n + \left(H_{[n/2]} + \frac{1 - (-1)^n}{n+1} \right) + \frac{2^{n+1} - 2}{n+1} \\ &= (2^{n+1} - 2)H_n + H_{[(n+1)/2]} + \frac{2^{n+1} - 2}{n+1}. \end{aligned}$$

This concludes the induction proof.

Proof of the congruences (4), (5), and (6). Using the identities $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$ ($1 \leq k \leq n$), $H_n = H_{n-1} + 1/n$ and the congruence (11) from Lemma 2.1, the left hand

side of (48) in Lemma 4.2 for $n = p$ is

$$\begin{aligned}
 & \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} 2^k H_k = \sum_{k=1}^{p-1} \frac{2^k p}{k} (-1)^{k-1} \binom{p-1}{k-1} \left(H_{k-1} + \frac{1}{k} \right) \\
 (54) \quad & \equiv \sum_{k=1}^{p-1} \frac{2^k p}{k} (1 - pH_{k-1}) \left(H_{k-1} + \frac{1}{k} \right) \pmod{p^3} \\
 & = p \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k} + p \sum_{k=1}^{p-1} \frac{2^k}{k^2} - p^2 \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} - p^2 \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k^2} \pmod{p^3}.
 \end{aligned}$$

Further, note that by (1) of Theorem 1.1, (19) of Lemma 2.3 and the identity $H_{k-1} = H_k - 1/k$,

$$(55) \quad \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k} = \sum_{k=1}^{p-1} \frac{2^k H_k}{k} - \sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -\frac{13p}{12} B_{p-3} \pmod{p^2}.$$

Taking (55), (42) and (19) of Lemma 2.3 into (54), we find that

$$\begin{aligned}
 (56) \quad & \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} 2^k H_k \equiv -pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 - \frac{7p^2}{6}B_{p-3} \\
 & - p^2 \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} \pmod{p^3}.
 \end{aligned}$$

On the other hand, by (48) of Lemma 4.2 with $n = p$,

$$(57) \quad \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} 2^k H_k = (2 - 2^p)H_{p-1} - H_{(p-1)/2} - 2q_p(2).$$

Furthermore, since by Wolstenholme's theorem and Fermat little theorem, $p^3 \mid H_{p-1}(2 - 2^p)$, taking this and (15) of Lemma 2.2 into (57) we get

$$(58) \quad \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} 2^k H_k \equiv -pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7p^2}{12}B_{p-3} \pmod{p^3}.$$

Now substituting (58) into (56), we obtain

$$p^2 \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} \equiv -\frac{7p^2}{4}B_{p-3} \pmod{p^3},$$

or equivalently,

$$(59) \quad \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} \equiv -\frac{7}{4}B_{p-3} \pmod{p}.$$

Finally, applying (59), (42) and (22) of Lemma 2.3, we have

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} &= \sum_{k=1}^{p-1} \frac{2^k \left(H_{k-1} + \frac{1}{k}\right)^2}{k} \\ &= \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} + 2 \sum_{k=1}^{p-1} \frac{2^k H_{k-1}}{k^2} + \sum_{k=1}^{p-1} \frac{2^k}{k^3} \\ &\equiv -\frac{1}{3} q_p(2)^3 + \frac{11}{24} B_{p-3} \pmod{p}. \end{aligned}$$

This is in fact the congruence (4).

In order to prove the congruence (5), notice that by (29) $H_{p-k} \equiv H_{k-1} \pmod{p}$ for each $k = 1, 2, \dots, p-1$. Hence, using this, the congruence (59), Fermat little theorem, and applying (4), (2) and (22), we find that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^2}{k \cdot 2^k} &= \sum_{k=1}^{p-1} \frac{H_{p-k}^2}{(p-k) \cdot 2^{p-k}} \equiv \sum_{k=1}^{p-1} \frac{H_{k-1}^2}{(-k) \cdot 2^{1-k}} \pmod{p} \\ &= -\frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k H_{k-1}^2}{k} = -\frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k \left(H_k - \frac{1}{k}\right)^2}{k} \\ &= -\frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} + \sum_{k=1}^{p-1} \frac{2^k H_k}{k^2} - \frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv \frac{7}{8} B_{p-3} \pmod{p}. \end{aligned}$$

This is in fact (5).

For establishing the congruence (6), observe that by (10) of Lemma 2.1,

$$p^2 H_{k,2} \equiv 2 - 2pH_k + p^2 H_k^2 - 2(-1)^k \binom{p-1}{k} \pmod{p^3},$$

whence we have

$$\begin{aligned} (60) \quad p^2 \sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} &\equiv 2 \sum_{k=1}^{p-1} \frac{2^k}{k} - 2p \sum_{k=1}^{p-1} \frac{2^k H_k}{k} + p^2 \sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} \\ &\quad - 2 \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p-1}{k} \pmod{p^3}. \end{aligned}$$

Finally, substituting the congruences (18) of Lemma 2.3, (1), (4) of Theorem 1.1 and (39) of Lemma 3.2 into (60), we obtain

$$p^2 \sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} \equiv -\frac{1}{3} p^2 q_p(2)^3 - \frac{25}{24} p^2 B_{p-3} \pmod{p^3}$$

from which we get

$$\sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} \equiv -\frac{1}{3} q_p(2)^3 - \frac{25}{24} B_{p-3} \pmod{p}.$$

This is the congruence (6), and the proof is completed. \square

5. PROOF OF COROLLARY 1.2

Lemma 5.1. *If $p > 3$ is a prime, then*

$$(61) \quad \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{ik^2} \equiv -\frac{5}{4} B_{p-3} \pmod{p},$$

$$(62) \quad \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{i^2 k} \equiv \frac{3}{4} B_{p-3} \pmod{p},$$

$$(63) \quad \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{ik} \equiv \frac{13}{12} p B_{p-3} \pmod{p^2}.$$

Proof. Since $H_{p-1} \equiv 0 \pmod{p^2}$ (the well known Wolstenholme's theorem), and thus for each $k = 1, 2, \dots, p-1$,

$$(64) \quad \sum_{i=k}^{p-1} \frac{1}{i} \equiv -\sum_{i=1}^{k-1} \frac{1}{i} = -H_{k-1} \pmod{p^2}.$$

Applying (64), (11) of Lemma 2.1 and taking the identity $\binom{p-1}{k-1} = \frac{k}{p} \binom{p}{k}$, we find that

$$\begin{aligned} p \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{ik^2} &= \sum_{k=1}^{p-1} \frac{2^k}{k^2} \sum_{i=k}^{p-1} \frac{p}{i} \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} p H_{k-1} \pmod{p^2} \\ &\equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \left(1 - (-1)^{k-1} \binom{p-1}{k-1} \right) \\ (65) \quad &= -\sum_{k=1}^{p-1} \frac{2^k}{k^2} - \sum_{k=1}^{p-1} \frac{(-2)^k}{k^2} \binom{p-1}{k-1} \\ &= -\frac{1}{p} \sum_{k=1}^{p-1} \frac{2^k}{k^2} - \sum_{k=1}^{p-1} \frac{(-2)^k}{k} \binom{p}{k}. \end{aligned}$$

Finally, taking the congruence (19) of Lemma 2.3 and (38) of Lemma 3.2 into the right hand side of (65), we immediately obtain (61).

Further, from (13) of Lemma 2.2 we see that $H_{p-1,2} \equiv 0 \pmod{p}$ and therefore, for each $k = 1, 2, \dots, p-1$,

$$\sum_{i=k}^{p-1} \frac{1}{i^2} \equiv -\sum_{i=1}^{k-1} \frac{1}{i^2} = -H_{k-1,2} \pmod{p}.$$

Applying this we obtain

$$(66) \quad \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{i^2 k} = \sum_{k=1}^{p-1} \frac{2^k}{k} \sum_{i=k}^{p-1} \frac{1}{i^2} \equiv -\sum_{k=1}^{p-1} \frac{2^k H_{k-1,2}}{k} \pmod{p}.$$

Further, taking $H_{k-1,2} = H_{k,2} - 1/k^2$, by (4) of Theorem 1.1 and (22) of Lemma 2.3, we get

$$(67) \quad \sum_{k=1}^{p-1} \frac{2^k H_{k-1,2}}{k} = \sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} - \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{3}{4} B_{p-3} \pmod{p}.$$

Inserting (67) into (66) we obtain (62).

Finally, by (64) we have

$$\begin{aligned} \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{ik} &= \sum_{k=1}^{p-1} \frac{2^k}{k} \sum_{i=k}^{p-1} \frac{1}{i} \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k} H_{k-1} \pmod{p^2} \\ &= -\sum_{k=1}^{p-1} \frac{2^k H_k}{k} + \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p^2} \end{aligned}$$

whence substituting the congruences (2) of Theorem 1.1 and (19) of Lemma 2.3, we obtain (63). \square

We are now ready to prove the congruence (7) from Corollary 1.2 conjectured by Z. W. Sun.

Proof of the congruence (7). Since $1/(p-k) \equiv -(p+k)/k^2 \pmod{p^2}$, we find that

$$\begin{aligned}
 2^p \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} &= 2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \sum_{i=1}^k \frac{1}{i} = 2^p \sum_{1 \leq i \leq k \leq p-1} \frac{1}{ik \cdot 2^k} \\
 &= 2^p \sum_{1 \leq p-i \leq p-k \leq p-1} \frac{1}{(p-i)(p-k)2^{p-k}} \\
 &\equiv \sum_{1 \leq k \leq i \leq p-1} \frac{(p+i)(p+k)2^k}{i^2 k^2} \pmod{p^2} \\
 (68) \quad &\equiv \sum_{1 \leq k \leq i \leq p-1} \frac{(pi + pk + ik)2^k}{i^2 k^2} \pmod{p^2} \\
 &= p \left(\sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{ik^2} + \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{i^2 k} \right) \\
 &\quad + \sum_{1 \leq k \leq i \leq p-1} \frac{2^k}{ik} \pmod{p^2}.
 \end{aligned}$$

The substitution of congruences (61)–(63) of Lemma 5.1 into (68) immediately yields

$$2^p \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \frac{7}{12} p B_{p-3} \pmod{p^2},$$

whence because of Fermat little theorem $2^{-p} \equiv 2^{-1} \pmod{p}$, we obtain

$$\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv 2^{-p} \frac{7}{12} p B_{p-3} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2},$$

as desired. □

Proof of the congruence (8). By (10) of Lemma 2.1,

$$p^2 H_{k,2} \equiv 2 - 2p H_k + p^2 H_k^2 - 2(-1)^k \binom{p-1}{k} \pmod{p^3},$$

whence we have

$$\begin{aligned}
 p^2 \sum_{k=1}^{p-1} \frac{H_{k,2}}{k \cdot 2^k} &\equiv 2 \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} - 2p \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} + p^2 \sum_{k=1}^{p-1} \frac{H_k^2}{k \cdot 2^k} \\
 (69) \quad &\quad - 2 \sum_{k=1}^{p-1} \frac{(-1)^k}{k \cdot 2^k} \binom{p-1}{k} \pmod{p^3}.
 \end{aligned}$$

Taking $n = p - 1$ and $x = 2$ into (37) from the proof of Lemma 3.1, we obtain

$$(70) \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k \cdot 2^k} \binom{p-1}{k} = \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} - H_{p-1}.$$

Substituting (70) into (69) yields

$$(71) \quad p^2 \sum_{k=1}^{p-1} \frac{H_{k,2}}{k \cdot 2^k} \equiv -2p \sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} + p^2 \sum_{k=1}^{p-1} \frac{H_k^2}{k \cdot 2^k} + 2H_{p-1} \pmod{p^3}.$$

Finally, substituting the congruences (7) of Corollary 1.2, (5) of Theorem 1.1 and (12) of Lemma 2.2 into (71), we obtain

$$p^2 \sum_{k=1}^{p-1} \frac{H_{k,2}}{k \cdot 2^k} \equiv -\frac{3}{8} p^2 B_{p-3} \pmod{p^3}$$

whence it follows that

$$\sum_{k=1}^{p-1} \frac{H_{k,2}}{k \cdot 2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p}.$$

This is the congruence (8), and the proof is completed. \square

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